



A THERMODYNAMICALLY CONSISTENT FORMULATION OF MAGNETOPOROELASTICITY

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Abstract—The dynamical theory of a porous, elastic, and magnetically inert solid body infiltrated by a ferromagnetic liquid is formulated. This type of magnetoporoelastic material opens a wide range of technical applications, such as controlled vibration dampers, sensors, and adaptive mirrors. The irreversible deformation and nonlinear coupling among the solid elastic, fluid rheological, and magnetic effects is derived based on the Biot–Frenkel theory with thermodynamic considerations. The thermal effect, however, is not considered in this paper. A linearized version of constitutive equations and the dynamic equations are introduced. © 1998 Elsevier Science Ltd. All rights reserved.

NOMENCLATURE

$A_f, A_s, A'_f, A'_s, C_f, C_s$	elastic constitutive constants
$a^{ss}, a^{sf}, a^{fs}, a^{fd}, a^{sd}$	elastic constitutive constants
$\mathbf{B}^f, B_i^f, \mathbf{B}^s, B_i^s$	magnetic induction of fluid and solid phase
b	resistivity coefficient
$b_k^s, b_k^f, b_{ij}^{fs}, b_{ij}^{ff}, b_{ij}^{fh}, b_{ijk}$	magnetic constitutive constants
$c^s, c^f, c_i^s, c_i^f, c$	coefficients for perturbed magnetic equation
e^s	solid internal volumetric strain
e_{ij}^s	solid internal strain tensor
$\langle e_{ij}^s \rangle$	solid internal deviatoric strain tensor
$\delta \mathcal{F}^f, \delta \mathcal{F}^s$	variation of free energy of the fluid and solid phase
F^f, F^s	volume averaged free energy density of fluid and solid phase
F_0^f, F_0^s	mechanical part of free energy density
f^f, f^s	free energy density of the fluid and solid phase
G_s	shear modulus of solid phase
g_i	gravity component
$\bar{\mathbf{H}}, \bar{H}_i$	average magnetic field of saturated porous medium
$\mathbf{H}^f, H_i^f, \mathbf{H}^s, H_i^s$	magnetic field of fluid and solid phase
I_i	interacting force density between phases
K_f, K_s	fluid and solid phase bulk modulus
h	thin layer thickness
$\mathbf{M}^f, \mathbf{M}^s$	magnetic moment per unit volume
m_H	magnetic moment of ferromagnetic particle
n, n_0	current and initial concentration of ferromagnetic particle
n_i, n_i^f, n_i^s	outward normal component of Σ, S^f and S^s
p^f, p^s	effective pressure in fluid and solid phase
p_H^f, p_H^s	magnetic part of effective pressure
p_m^f, p_m^s	mechanical part of effective pressure
Q^f, Q^s, Q	fluid, solid and total average force due to surface stresses
R	coefficient for magnetic susceptibility
S^f	surface of internal pore space
S^f, S^s	bounding surface of Ω^f, Ω^s
t_i^f, t_i^s	fluid and solid surface traction component
$\delta U_i^f, \delta U_i^s$	external surface averaged fluid and solid virtual displacement

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$\delta u_i^f, \delta u_i^s$	volume averaged fluid and solid virtual displacement
$\delta \mathcal{W}^f, \delta \mathcal{W}^s$	virtual work of the fluid and solid phase
$\delta W^f, \delta W^s$	virtual work density of the fluid and solid phase

Greek symbols

β_f, β_s	fluid and solid phase magnetic field weighing factor
ϵ^f, ϵ^s	fluid and solid external volumetric strain
$\epsilon_{ij}^f, \epsilon_{ij}^s$	fluid and solid external strain tensor
$\langle \epsilon_{ij}^f \rangle, \langle \epsilon_{ij}^s \rangle$	fluid and solid external deviatoric strain tensor
ζ_i^f, ζ_i^s	fluid and solid virtual displacement
η	magnetic field partition factor
μ^f, μ^s	magnetic permeability of fluid and solid phase
ν	fluid kinematic viscosity
ν_d	fluid dynamic viscosity
ξ_i^f, ξ_i^s	external linear strain component of fluid and solid phase
$\rho_f, \rho_s, \rho, \rho_d$	fluid, solid, bulk and added mass density
$\sigma_{ij}^f, \sigma_{ij}^s, \sigma_{ij}$	fluid, solid and total stress tensor
ϕ	porosity
χ	magnetic susceptibility
Σ	bounding surface of Ω
Σ^f, Σ^s	external surface of fluid and solid phase
$\Omega, \Omega^f, \Omega^s$	volume of total sample, fluid and solid matrix
(~)	to denote perturbed quantities

1. INTRODUCTION

Colloid solutions and stabilized emulsions of small-size ferromagnetic particles having high values of magnetic moment are finding a wide range of applications in modern technology, such as active optical surfaces, mechanical manipulators, actively controlled vibration dampers, etc. (Ehrgott and Marsi, 1992; Fertman, 1988; Papell, 1965; Takeomi and Tickatsumi, 1988; Shliomis *et al.*, 1988). However, the fact that such materials are liquid has limited their even broader usage. In many cases better effect may be achieved by using non-liquid and highly elastic materials. A solid material can provide a support surface, yet can still significantly change its volume and form when exposed to an external magnetic field. One possibility to realize such a kind of magneto-active composite material is to organize it as a magnetically relatively inert solid matrix containing magnetic inclusions, such as the magnetic rubber. This type of material, however, exhibits relatively low magnetic permeability as for non-connected inclusions the magnetic permeability is dominated by that of the inert matrix, not of the inclusion.

Yet another innovative idea is to create a material that is made of a porous, elastic solid matrix, which is magnetically relatively inert, but is infiltrated with a ferromagnetic liquid (Lopatnikov, 1997). The major physical difference between such solid material with inclusion and the proposed magnetoporoelastic material is that the high magnetic phase in the latter case is continuous and can conduct a magnetic field, as opposed to a solid case of isolated inclusions. A porous penetrable material can have a magnetic constant close to that of a ferromagnetic fluid.

In contrast to the magnetorheological liquids, the magnetoporoelastic materials have the mechanical properties of a solid. They can retain their form for an indefinite amount of time if the surface pores are closed, or pore size is small enough and the associated surface tension is high enough to prevent flow. On the other hand, the body may be highly deformable if a relative movement of the magnetic and non-magnetic phases is allowed on the surface. Estimates show that a magnetoporoelastic material can be prepared so that under moderate magnetic fields the deformation can reach to about 1–10% of its thickness (Lopatnikov, 1997).

The dynamic properties of the magnetorheological liquid saturated porous media are also unique. For instance, at the low frequency range, a body can be excited by the dynamic of viscous pore liquid under the pressure induced by external magnetic field. Also, it is possible to show that the sound velocity and dissipation of acoustic waves depend on the applied external field.

These properties suggest that these materials are open for a wide range of applications. They can be useful, for instance, as the active bodies of devices of adaptive optics (for

instance, for the manufacturing of magnetically controlled mirrors); as controlled dampers of vibration; for controlled filters of a gas or liquids, etc.

There have been a fair number of theoretical investigations on ferromagnetic liquids, and relatively few in solid composites with ferromagnetic inclusions. To our knowledge, the theoretical study of porous elastic materials saturated with ferromagnetic liquids has not been conducted. It is the purpose of this paper to provide a formulation of the dynamic behavior of magnetoporoelastic materials based on the Biot–Frenkel theory (Biot, 1941, 1955, 1956a, b, 1962; Frenkel, 1944).

Following thermodynamic considerations, we present the free energies of solid and fluid phases with mechanical and magnetic origins. Thermal energy is not included in the present effort. By the virtual displacement program, irreversibility and nonlinearity due to porosity variation emerges. Stresses are identified as those from mechanical or magnetic origins, or from the coupling of the two. For the magnetic field, a simple model of effective magnetic permeability for the composite material is proposed. Constitutive and dynamic equilibrium equations are derived. These equations are then linearized to provide some aspects of the linear dynamic behavior of such bodies.

2. VIRTUAL WORK

First of all we need to generalize the theory of poroelasticity to include the effect of a ferromagnetic fluid and its interaction with a magnetic field. The most appropriate approach is to consider the thermodynamics of deformation of a composite material using the well known principle of virtual work to account for the energy transfer between the phases.

For a representative elementary volume (REV) of porous medium saturated with fluid, let Ω be the volume of the entire sample, Ω^s be the volume of the solid matrix, and Ω^f be the fluid part (see Fig. 1). It is clear that $\Omega = \Omega^s + \Omega^f$. The porosity of the sample is defined as $\phi = \Omega^f/\Omega$. The portion occupied by the solid is obviously $\Omega^s/\Omega = 1 - \phi$. Let Σ denote the boundary of Ω , S^f the boundary of Ω^f , and S^s the boundary of Ω^s . The intersection $S^i = S^f \cap S^s$ represents the surface of internal pore space. Then $\Sigma^s = S^s \setminus S^i$ is the external part of the solid phase surface and $\Sigma^f = S^f \setminus S^i$ is the external part fluid phase surface. It is clear that $\Sigma^s \cup \Sigma^f = \Sigma$. We shall also assume that volumetric porosity is equal to areal porosity, namely, $\Sigma^f/\Sigma = \Omega^f/\Omega = \phi$ and $\Sigma^s/\Sigma = 1 - \phi$.

Consider a sample of the porous medium having porosity ϕ which is deforming under the surface traction t_i^s and t_i^f , respectively, acting on S^s and S^f . These traction forces can be expressed as $t_i^s = \sigma_{ij}^s n_j^s$ and $t_i^f = \sigma_{ij}^f n_j^f$, where σ_{ij}^s and σ_{ij}^f are, respectively, the solid and fluid stress tensor, and n_j^s and n_j^f are components of the outward normal vector of S^s and S^f , respectively. The virtual work done by the surface traction to each phase is, therefore,

$$\delta \mathcal{W}^s = \int_{S^s} \sigma_{ij}^s n_j^s \delta \zeta_i^s \, dS \tag{1}$$

$$\delta \mathcal{W}^f = \int_{S^f} \sigma_{ij}^f n_j^f \delta \zeta_i^f \, dS \tag{2}$$

where $\delta \mathcal{W}^s$ and $\delta \mathcal{W}^f$ are virtual work, respectively, of the solid and the fluid phase, $\delta \zeta_i^s$ and $\delta \zeta_i^f$ are solid and fluid virtual displacements, respectively. Within each phase, we further decompose the work into those acting on the internal surface, $\delta \mathcal{W}_{\text{int}}^s$, and those on external surface, $\delta \mathcal{W}_{\text{ext}}^s$:

$$\begin{aligned} \delta \mathcal{W}^s &= \delta \mathcal{W}_{\text{ext}}^s + \delta \mathcal{W}_{\text{int}}^s \\ &= \int_{\Sigma^s} \sigma_{ij}^s n_j^s \delta \zeta_i^s \, dS + \int_{S^i} \sigma_{ij}^s n_j^s \delta \zeta_i^s \, dS \end{aligned} \tag{3}$$

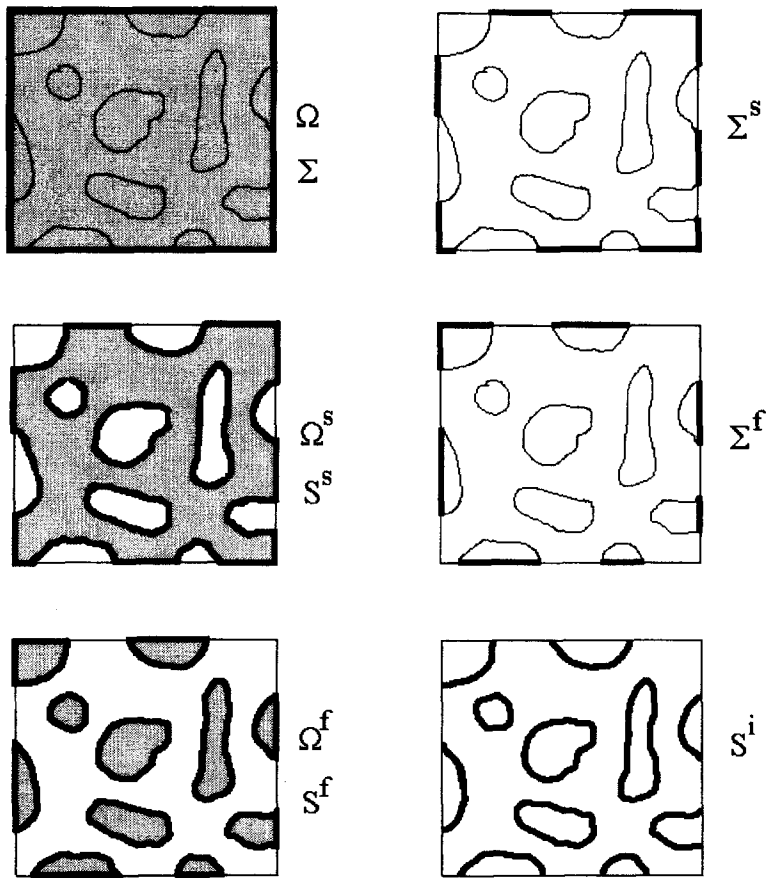


Fig. 1. Illustration of various domains and boundaries of solid and fluid phases.

$$\begin{aligned} \delta \mathcal{W}^f &= \delta \mathcal{W}_{\text{ext}}^f + \delta \mathcal{W}_{\text{int}}^f \\ &= \int_{\Sigma^f} \sigma_{ij}^f n_j \delta \zeta_i^f \, dS + \int_{S^f} \sigma_{ij}^f n_j^f \delta \zeta_i^f \, dS \end{aligned} \tag{4}$$

In the above we note that $n_j^s = n_j^f = n_j$, on the external surface Σ^s and Σ^f .

To simplify the derivation, but without the loss of generality, we shall follow the classical approach of Landau and Lifshitz (1982) (Sections 15 and 16). We consider the REV as a thin, flat layer. This assumption reduces the dimensionality and simplifies the presentation, yet without changing the outcome in a multi-dimensional analysis. This planar layer has a constant thickness h and constant material properties. We assume that the surface is traction and body force on and in the REV are uniform. We subject the REV to certain infinitesimal virtual displacements of solid and fluid, $\delta \zeta_i^s$ and $\delta \zeta_i^f$, whose directions are arbitrary and need not to be that of the outward normal of the plane, \mathbf{n} .

Under these conditions, the work done by forces on the external surfaces, as defined in (3) and (4), can be written as :

$$\delta \mathcal{W}_{\text{ext}}^s = \sigma_{ij}^s n_j \int_{\Sigma^s} \delta \zeta_i^s \, dS = (1 - \phi) \Sigma_0 \sigma_{ij}^s n_j \delta U_i^s \tag{5}$$

$$\delta \mathcal{W}_{\text{ext}}^f = \sigma_{ij}^f n_j \int_{\Sigma^f} \delta \zeta_i^f \, dS = \phi \Sigma_0 \sigma_{ij}^f n_j \delta U_i^f \tag{6}$$

where

$$\delta U_i^s = \frac{1}{\Sigma_0^s} \int_{\Sigma^s} \delta \zeta_i^s \, dS \tag{7}$$

$$\delta U_i^f = \frac{1}{\Sigma_0^f} \int_{\Sigma^f} \delta \zeta_i^f \, dS \tag{8}$$

are the averaged displacements of each phase over the external surface of the REV (Fig. 1), ϕ is the porosity, Σ_0 is the initial total external surface, $\Sigma_0^s = (1 - \phi)\Sigma_0$ and $\Sigma_0^f = \phi\Sigma_0$. For work done on the internal surfaces, as defined in (3) and (4), we notice that $\sigma_{ij}^s = \sigma_{ij}^f$, $\delta \zeta_i^s = \delta \zeta_i^f$, yet $n_j^s = -n_j^f$. It is clear that

$$\delta \mathcal{W}_{\text{int}}^s + \delta \mathcal{W}_{\text{int}}^f = 0 \tag{9}$$

3. ENERGY BALANCE

From energy conservation, the variation of work is equal to the variation of free energy

$$\delta \mathcal{W}^s = \delta \mathcal{F}^s \tag{10}$$

$$\delta \mathcal{W}^f = \delta \mathcal{F}^f \tag{11}$$

where \mathcal{F}^s and \mathcal{F}^f are, respectively, free energy of the solid and the fluid phase. Using f to denote the free energy per unit volume, we have, within the first order of approximation, the variation of free energy as

$$\delta \mathcal{F}^s = \delta \int_{\Omega^s} f^s \, d\Omega = F^s \delta \Omega^s + \Omega_0^s \delta F^s \tag{12}$$

$$\delta \mathcal{F}^f = \delta \int_{\Omega^f} f^f \, d\Omega = F^f \delta \Omega^f + \Omega_0^f \delta F^f \tag{13}$$

where F is the volume averaged free energy density given as

$$F^s = \frac{1}{\Omega_0^s} \int_{\Omega^s} f^s \, d\Omega \tag{14}$$

$$F^f = \frac{1}{\Omega_0^f} \int_{\Omega^f} f^f \, d\Omega \tag{15}$$

where Ω_0^s and Ω_0^f are, respectively, the initial undeformed solid and fluid volume.

Following the definitions in (3), (4), (10) and (11), the energy equations can be expressed as

$$\delta \mathcal{W}_{\text{ext}}^s - \delta \mathcal{F}^s = -\delta \mathcal{W}_{\text{int}}^s \tag{16}$$

$$\delta \mathcal{W}_{\text{ext}}^f - \delta \mathcal{F}^f = -\delta \mathcal{W}_{\text{int}}^f \tag{17}$$

Using (12) and (13) in the above, we obtain

$$\delta \mathcal{W}_{\text{ext}}^s - F^s \delta \Omega^s - \Omega_0^s \delta F^s = -\delta \mathcal{W}_{\text{int}}^s \tag{18}$$

$$\delta \mathcal{W}_{\text{ext}}^f - F^f \delta \Omega^f - \Omega_0^f \delta F^f = -\delta \mathcal{W}_{\text{int}}^f \tag{19}$$

Through the variation of solid and fluid volume terms, $\delta\Omega^s$ and $\delta\Omega^f$, the role of porosity variation is introduced :

$$\delta\Omega^s = \delta[(1 - \phi)\Omega] = (1 - \phi)\delta\Omega - \Omega_0\delta\phi \tag{20}$$

$$\delta\Omega^f = \delta(\phi\Omega) = \phi\delta\Omega + \Omega_0\delta\phi \tag{21}$$

where Ω_0 is the initial total volume. Referring to the “thin-layer” of thickness h , we may further express

$$\delta\Omega = \delta U_n^s \Sigma_0 = \frac{\delta U_n^s \Omega_0}{h} \tag{22}$$

in which $\delta U_n^s = \delta U_i^s n_i$ is the normal component of δU_i^s vector, defined in (7), and $\Omega_0 = \Sigma_0 h$. Here we introduce the notation of solid linear strain based on external virtual displacement as

$$\delta \xi_i^s = \frac{\delta U_i^s}{h} \tag{23}$$

and its normal component $\xi_n^s = \xi_i^s n_i$. Using (22) and (23), we can rewrite (20) as

$$\delta\Omega^s = \Omega_0[(1 - \phi)\delta\xi_n^s - \delta\phi] \tag{24}$$

This shows that the change of solid volume contains a solid strain part and a porosity variation part. By the same token, the fluid volume change is

$$\delta\Omega^f = \Omega_0[\phi\delta\xi_n^f + \delta\phi] \tag{25}$$

where $\delta\xi_n^f = \delta U_n^f/h$ is the fluid normal strain.

Substituting (5), (6), (24), (25) and (18) and (19), and dividing the equations by Ω_0 , we find

$$(1 - \phi)(\sigma_{ij}^s \delta \xi_i^s n_j - \delta \xi_n^s F^s - \delta F^s) + F^s \delta \phi = -\delta W_{int}^s \tag{26}$$

$$\phi(\sigma_{ij}^f \delta \xi_i^f n_j - \delta \xi_n^f F^f - \delta F^f) - F^f \delta \phi = -\delta W_{int}^f \tag{27}$$

where $W_{int} = \mathcal{W}_{int}/\Omega_0$ is the density of work done by forces on the pore surface. If the two equations are summed, the energy balance of the combined solid and fluid phase is

$$(1 - \phi)(\sigma_{ij}^s \delta \xi_i^s n_j - \delta \xi_n^s F^s - \delta F^s) + \phi(\sigma_{ij}^f \delta \xi_i^f n_j - \delta \xi_n^f F^f - \delta F^f) + (F^s - F^f)\delta\phi = 0 \tag{28}$$

To generalize the result from a thin layer to a three-dimension body, it is convenient to introduce the notation of external strain tensor as

$$\delta \varepsilon_{ij}^s = \frac{1}{2} (\delta \xi_i^s n_j + \delta \xi_j^s n_i) \tag{29}$$

$$\delta \varepsilon_{ij}^f = \frac{1}{2} (\delta \xi_i^f n_j + \delta \xi_j^f n_i) \tag{30}$$

It is clear that

$$\delta \zeta_n^s = \delta \varepsilon_{ii}^s = \delta \varepsilon^s \tag{31}$$

$$\delta \zeta_n^f = \delta \varepsilon_{ii}^f = \delta \varepsilon^f \tag{32}$$

are the volumetric strains. Equation (28) then becomes

$$(1 - \phi)(\sigma_{ij}^s \delta \varepsilon_{ij}^s - F^s \delta \varepsilon^s - \delta F^s) + \phi(\sigma_{ij}^f \delta \varepsilon_{ij}^f - F^f \delta \varepsilon^f - \delta F^f) + (F^s - F^f) \delta \phi = 0 \tag{33}$$

We may consider (33) as the law of conservation of total energy for porous media. Equation (33) differs from earlier results (Biot, 1941, 1955; Nikolaevskiy, 1984, 1996) in that it contains a porosity variation term. This term allows the modeling of the structural change of porous media which is generally an irreversible process. Its modeling based on the virtual displacement approach will be discussed later. It is also important to note that the virtual work associated with the internal surface does not appear in (33). It is possible to eliminate the $\delta \phi$ term between (26) and (27) such that the internal work term δW_{int}^s appears in the energy equation. In that case, a model for δW_{int}^s must be provided.

4. VARIATION OF FREE ENERGY

We next consider the variation of free energy. For convenience, we shall consider only the free energy due to the elastic and the magnetic fields, and shall ignore heat energy and energy of other origins. We also restrict the attention to isotropic elastic medium. We hence write

$$F^s = F^s(e_{ij}^s, \mathbf{H}^s) \tag{34}$$

$$F^f = F^f(\rho_f, \mathbf{H}^f) \tag{35}$$

where \mathbf{H}^s and \mathbf{H}^f are the microscopically averaged magnetic fields for solid and fluid phase, respectively. Here we have introduced the internal strain e_{ij}^s which is defined from the volume averaged virtual displacements

$$\delta u_i^s = \frac{1}{\Omega_0^s} \int_{\Omega'} \delta \zeta_i^s d\Omega \tag{36}$$

$$\delta u_i^f = \frac{1}{\Omega_0^f} \int_{\Omega'} \delta \zeta_i^f d\Omega \tag{37}$$

and

$$\delta e_{ij}^s = \frac{1}{2} (\delta u_{i,j}^s + \delta u_{j,i}^s) \tag{38}$$

Equations (36) and (37) should be compared with (7) and (8), as the former pair are average over the volume of the solid and fluid phases, while the latter are averaged over the surface of REV. Hence a distinction is made between the volume averaged “internal” strains e_{ij}^s and e_{ij}^f as needed in defining the free energy, and the external surface average “external” strains ε_{ij}^s and ε_{ij}^f associated with the virtual work. The solid external strain ε_{ij}^s is also the apparent strain of the REV as observed from the external surface of the volume. The internal and external strains are not equated by a trivial relation due to the possible presence of porosity variation attributed to the internal rebuilding of pore structure. We also notice that for the fluid part, the fluid density ρ_f is used as the variational parameter in lieu of fluid strain because a significantly larger deformation for fluid (such as gas) than that for solid will be considered. The use of fluid density will allow the introduction of the

equation of state for fluid when necessary. Utilizing (24) and (25), we can show that the volumetric strains and density variation are related to porosity variation as

$$\delta e^s = \frac{\delta \Omega^s}{\Omega_0^s} = \delta \varepsilon^s - \frac{\delta \phi}{1 - \phi} \quad (39)$$

$$\frac{\delta \rho_f}{\rho_f} = - \frac{\delta \Omega^f}{\Omega_0^f} = - \delta \varepsilon^f - \frac{\delta \phi}{\phi} \quad (40)$$

in which we denote $\delta e^s = \delta e_{ii}^s$. It is clear that only in the absence of porosity variation, the internal and external strains are equal.

The functional relationships in (34) and (35) allow us to write

$$\delta F^s = \left(\frac{\partial F^s}{\partial e^s} \right)_{T, \mathbf{H}} \delta e^s + \left(\frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} \right)_{T, \mathbf{H}} \langle \delta e_{ij}^s \rangle + \left(\frac{\partial F^s}{\partial \mathbf{H}^s} \right)_{T, e_{ij}^s} \delta \mathbf{H}^s \quad (41)$$

$$\delta F^f = \left(\frac{\partial F^f}{\partial \rho_f} \right)_{T, \mathbf{H}} \delta \rho_f + \left(\frac{\partial F^f}{\partial \mathbf{H}^f} \right)_{T, \rho_f} \delta \mathbf{H}^f \quad (42)$$

in which we have separated e_{ij}^s into a volumetric part, e^s , and a deviatoric part,

$$\langle \delta e_{ij}^s \rangle = \delta e_{ij}^s - \frac{1}{3} \delta_{ij} \delta e^s \quad (43)$$

We note that not all of the strain components are independent. For example, for an isotropic material, there exist only two quadratic invariants $e^s \cdot e^s$ and $\langle e_{ij}^s \rangle \langle e_{ij}^s \rangle$ (Einstein summation of repeated indices is implied throughout the paper) that the free energy is dependent upon. In (41) and (42) the delimitation with subscript T is used to emphasize the isothermal nature of the process and shall be dropped henceforth. By definition (Landau and Lifshitz, 1982), the magnetic induction vector (magnetic displacement), \mathbf{B} is the derivative of free energy with respect to magnetic field, \mathbf{H} ,

$$\left(\frac{\partial F^s}{\partial \mathbf{H}^s} \right)_{e_{ij}^s} = - \frac{1}{4\pi} \mathbf{B}^s \quad (44)$$

$$\left(\frac{\partial F^f}{\partial \mathbf{H}^f} \right)_{\rho_f} = - \frac{1}{4\pi} \mathbf{B}^f \quad (45)$$

In the above \mathbf{B}^s and \mathbf{B}^f are the magnetic induction of solid and fluid phase, respectively. Substituting (39)–(45) into (33), we obtain:

$$\begin{aligned} (1 - \phi) \left\{ \left[\sigma_{ij}^s - \left(F^s + \frac{\partial F^s}{\partial e^s} \right) \delta_{ij} \right] \delta e_{ij}^s - \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} \langle \delta e_{ij}^s \rangle + \frac{1}{4\pi} B_i^s \delta H_i^s \right\} \\ + \phi \left\{ \left[\sigma_{ij}^f - \left(F^f - \rho_f \frac{\partial F^f}{\partial \rho_f} \right) \delta_{ij} \right] \delta \varepsilon_{ij}^f + \frac{1}{4\pi} B_i^f \delta H_i^f \right\} + \left[\left(F^s + \frac{\partial F^s}{\partial e^s} \right) - \left(F^f - \rho_f \frac{\partial F^f}{\partial \rho_f} \right) \right] \delta \phi = 0 \end{aligned} \quad (46)$$

where B_i and H_i are Cartesian components of \mathbf{B} and \mathbf{H} .

5. VARIATION OF POROSITY

Equation (46) shows that, in the general case, there is an internal degree of freedom associated with possible space redistribution of the material. This part of deformation is responsible for the irreversible work energy of the system. In the present case, this degree of freedom is described by the variation of porosity only.

In the special case that the deformation involves only a pure rebuilding of the porous medium and there is no volumetric deformation of the solid and fluid phases, namely,

$$\delta\rho_f = \delta e^s = 0 \tag{47}$$

(39) and (40) show that the external strains δe^s and δe^f are directly a consequence of porosity variation. This reduces the degree of freedom of (46). In the absence of magnetic field, the variation of total energy as well as the work done by external forces along this deformation path ($\delta e^s, \delta e^f$) are equal to zero. It means that the sample is at its initial quasi-neutral equilibrium and can change its structure under arbitrarily small action. This case applies to an unconsolidated medium.

On the other hand, if a consolidated medium is in a stable initial geometric configuration, the variation of energy will not be zero under all trajectories of ($\delta e^s, \delta e^f$). In that case, the energy surface in ($\delta e^s, \delta e^f$) space must have a minimum and the associated Hessian matrix must be negative. It means that one cannot rebuild the internal structure of a porous medium without changing the energy in the phases. These conditions are satisfied if there exists some additional relation between $\delta\phi$ and other state variables, such as $\delta\rho_f$ and δe^s , or δe^s and δe^f . For example, we may write

$$\delta\phi = \phi(1-\phi) \left[A'_s(\rho_f, e^s_{ij}, \mathbf{H}^s, \mathbf{H}^f)\delta e^s - A'_f(\rho_f, e^s_{ij}, \mathbf{H}^s, \mathbf{H}^f) \frac{\delta\rho_f}{\rho_f} \right] \tag{48}$$

In the above, A'_s and A'_f are constitutive coefficients, and the factor $\phi(1-\phi)$ is introduced for convenience of obtaining a more symmetric form in subsequent derivation. Since we assume that the mechanical variations take place under stationary magnetic field, the magnetic field variables enter the relations only through the constitutive coefficients. Alternatively, (48) can be expressed as

$$\delta\phi = \phi(1-\phi)[A_s(\rho_f, e^s_{ij}, \mathbf{H}^s, \mathbf{H}^f)\delta e^s + A_f(\rho_f, e^s_{ij}, \mathbf{H}^s, \mathbf{H}^f)\delta e^f] \tag{49}$$

The deformation of porous medium is a reversible process only if $\delta\phi$ defined either in (48) or in (49) is a complete differential. This will impose limitation on the form of coefficients A_s and A_f (or A'_s and A'_f). Hence, generally speaking, the deformation is irreversible.

From (39), (40) and (49) we can solve for

$$\delta e^s = (1-\phi A_s)\delta e^s - \phi A_f \delta e^f \tag{50}$$

$$\frac{\delta\rho_f}{\rho_f} = -(1-\phi)A_s\delta e^s - (1+A_f-\phi A_f)\delta e^f \tag{51}$$

We can consider (50) and (51) as differential relations between the internal thermodynamic mechanical parameters and external parameters. Using (50) and (51) in (49) we obtain

$$\delta\phi = \phi(1-\phi) \frac{A_s\delta e^s - A_f(\delta\rho_f/\rho_f)}{(1-\phi A_s) + (1-\phi)A_f} \tag{52}$$

Comparing with (48), this shows that A_s, A_f, A'_s and A'_f are related as

$$A'_s = \frac{A_s}{(1-\phi)A_s + (1-\phi)A_f} \quad (53)$$

$$A'_f = \frac{A_f}{(1-\phi)A_s + (1-\phi)A_f} \quad (54)$$

The constitutive relations introduced so far involve volumetric strains. For isotropic elastic material, a relation between the external and internal deviatoric strain needs to be established. The constitutive relation is given by

$$\langle \delta e_{ij}^s \rangle = C_s(\rho_f, e_{ij}^s, \mathbf{H}^s, \mathbf{H}^f) \langle \delta \varepsilon_{ij}^s \rangle + C_f(\rho_f, e_{ij}^s, \mathbf{H}^s, \mathbf{H}^f) \langle \delta \varepsilon_{ij}^f \rangle \quad (55)$$

where C_s and C_f are constitutive constants, and $\langle \delta \varepsilon_{ij}^s \rangle = \delta \varepsilon_{ij}^s - \frac{1}{3} \delta_{ij} \delta \varepsilon^s$ and $\langle \delta \varepsilon_{ij}^f \rangle = \delta \varepsilon_{ij}^f - \frac{1}{3} \delta_{ij} \delta \varepsilon^f$ are deviatoric external strains of solid and fluid, respectively. Here we note that a Newtonian fluid does not resist shear, hence $C_f = 0$. In general, however, the dynamic and rheological effects can come into play. The coefficient C_f can not only be a constant, but also take the form of a time operator, such as

$$C_f = \tau \frac{d}{dt} \quad (56)$$

where τ is a characteristic time. There exist several characteristic times. One is associated with the Biot characteristic frequency, and a second is related to the ratio of fluid and solid shear properties (viscosity and shear modulus). For dynamic excitation around these characteristic time ranges, their effect becomes important.

6. STRESS TENSOR

We now examine the terms δH_i^s and δH_i^f in (46). The magnetic field is governed by the equation

$$\nabla \times \mathbf{H} = \mathbf{0} \quad (57)$$

which implies that it can be written as the gradient of a potential (with a negative sign)

$$\mathbf{H} = -\nabla \varphi \quad (58)$$

We can write

$$\delta \mathbf{H} = -\delta(\nabla \varphi) = -\nabla(\delta \varphi) \quad (59)$$

Under a virtual displacement, we assume that each particle carries the potential with it. The change of potential at a fixed point can be expressed by the Eulerian derivative

$$\delta \varphi = -\delta \mathbf{u} \cdot \nabla \varphi = \delta \mathbf{u} \cdot \mathbf{H} \quad (60)$$

Utilizing the thin-layer approach, we assume that the magnetic field \mathbf{H} and the strain $\delta \xi$ are uniform in the layer. The virtual displacement can, therefore, be written as a linear function of z (perpendicular to the layer)

$$\delta \mathbf{u} = \delta \xi z \quad (61)$$

Substituting (60) and (61) into (59), we find

$$\delta \mathbf{H} = -\nabla(\mathbf{H} \cdot \delta \xi z) = -(\mathbf{H} \cdot \delta \xi) \nabla z \quad (62)$$

Here we note the constancy of \mathbf{H} and $\delta \xi$. Since ∇z is in the \mathbf{n} direction, we can write (62) in the general coordinate system for solid and fluid phases as (cf. Landau and Lifshitz, 1982 Section 15)

$$\delta H_i^s = -H_j^s \delta e_{ij}^s \quad (63)$$

$$\delta H_i^f = -H_j^f \delta e_{ij}^f \quad (64)$$

Next, we introduce the effective pressure due to mechanical deformation in the solid and fluid phase as

$$p^s(e^s, \langle e_{ij}^s \rangle, \mathbf{H}^s) = -\left(F^s + \frac{\partial F^s}{\partial e^s}\right) \quad (65)$$

$$p^f(\rho_f, \mathbf{H}^f) = -\left(F^f - \rho_f \frac{\partial F^f}{\partial \rho_f}\right) \quad (66)$$

In the absence of a magnetic field, (65) and (66) are the usual thermodynamic definition of pressure.

Substitution of (49), (55) and (63)–(66) into (46) yields the following

$$(1-\phi) \left\{ \sigma_{ij}^s + [p^s - \phi A_s(p^s - p^f)] \delta_{ij} - C_s \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} - \frac{1}{4\pi} B_i^s H_j^s \right\} \delta e_{ij}^s + \phi \left\{ \sigma_{ij}^f + [p^f - (1-\phi) A_f(p^s - p^f)] \delta_{ij} - \frac{1-\phi}{\phi} C_f \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} - \frac{1}{4\pi} B_i^f H_j^f \right\} \delta e_{ij}^f = 0 \quad (67)$$

Because all virtual displacements are independent, we have

$$\sigma_{ij}^s = \frac{1}{4\pi} B_i^s H_j^s - \delta_{ij} [p^s - \phi A_s(p^s - p^f)] + C_s \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} \quad (68)$$

$$\sigma_{ij}^f = \frac{1}{4\pi} B_i^f H_j^f - \delta_{ij} [p^f - (1-\phi) A_f(p^s - p^f)] + \frac{1-\phi}{\phi} C_f \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} \quad (69)$$

Now we have completed the program of virtual displacement. Equations (68) and (69) define the stress tensor in the solid and fluid phases. The first term on the right hand side is a magnetic stress. The pressure terms, as will be demonstrated in the next section, contains a mechanical part and a magnetic part. It is important to point out that these equations are valid, in general, for arbitrary equations of state of fluid, solid and magnetic field, as opposed to equations suggested in earlier works (Biot, 1941, 1955; Nikolaevskiy, 1984, 1996). Some specific cases will be investigated in the following.

7. MAGNETIC PRESSURE

Let us consider (65) and (66) in more detail by separating the effects of mechanic and magnetic fields. The free energy of a ferromagnetic liquid can be expressed as (Landau and Lifshitz, 1982; Takeomi and Tickatsumi, 1988)

$$F^f(\rho_f, \mathbf{H}^f) = F_0^f(\rho_f) - \frac{1}{4\pi} \int_0^{\mathbf{H}^f} \mathbf{B}^f(\mathbf{H}^f) \cdot d\mathbf{H}^f \quad (70)$$

where F_0^f is the mechanical free energy (in the absence of a magnetic field). It is well known from thermodynamics that the derivative of free energy per unit mass with respect to specific volume is the pressure (with a negative sign) :

$$p_m^f(\rho_f) = - \frac{\partial}{\partial(1/\rho_f)} \left(\frac{F_0^f}{\rho_f} \right) = - \left(F_0^f - \rho_f \frac{\partial F_0^f}{\partial \rho_f} \right) \quad (71)$$

Using (70) and (71), (66) can be expressed as

$$p^f = p_m^f + p_H^f \quad (72)$$

in which we have analogously defined a magnetic pressure which is associated with the magnetic part of free energy under mechanical variation :

$$\begin{aligned} p_H^f &= \frac{\partial}{\partial(1/\rho_f)} \left(\frac{1}{4\pi\rho_f} \int_0^{\mathbf{H}^f} \mathbf{B}^f \cdot d\mathbf{H}^f \right) \\ &= \frac{1}{4\pi} \left(\int_0^{\mathbf{H}^f} \mathbf{B}^f \cdot d\mathbf{H}^f - \rho_f \frac{\partial \int_0^{\mathbf{H}^f} \mathbf{B}^f \cdot d\mathbf{H}^f}{\partial \rho_f} \right) \end{aligned} \quad (73)$$

The magnetic induction vector can be written in this form

$$\mathbf{B}^f = \mathbf{H}^f + 4\pi\mathbf{M}^f \quad (74)$$

where \mathbf{M}^f , known as the magnetization vector, is the magnetic moment per unit volume. In the present case we note that the ferromagnetic liquid is an emulsion made of ordinary liquid containing suspended nano-size ferromagnetic particles. Each of the particles is an elementary magnet. The magnetic moment density \mathbf{M}^f is averaged over an REV and is dependent on the concentration of the particles. The temperature dependence is also present because the disordering of the individual magnetic moments cancels out the field. This effect, however, is not explicitly considered here. Using (74), we can present (73) as

$$p_H^f = \int_0^{\mathbf{H}^f} \frac{\partial(\mathbf{M}^f/\rho_f)}{\partial(1/\rho_f)} \cdot d\mathbf{H}^f + \frac{1}{8\pi} \mathbf{H}^f \cdot \mathbf{H}^f \quad (75)$$

By analogy, we can develop the equations for solid phase. The pressure is partitioned into two parts

$$p^s = p_m^s + p_H^s \quad (76)$$

where

$$p_m^s = - \left(F_0^s + \frac{\partial F_0^s}{\partial e^s} \right) \quad (77)$$

is the partial pressure from the change of mechanical free energy under mechanical virtual displacement, and

$$p_H^s = \int_0^{\mathbf{H}^s} \mathbf{M}^s \cdot d\mathbf{H}^s + \int_0^{\mathbf{H}^s} \frac{\partial \mathbf{M}^s}{\partial e^s} \cdot d\mathbf{H}^s + \frac{1}{8\pi} \mathbf{H}^s \cdot \mathbf{H}^s \tag{78}$$

is the partial pressure from the change of magnetic free energy under a mechanical virtual displacement.

8. EFFECTIVE MAGNETIC PERMEABILITY

The next problem we face is to establish the relationship among the external (apparent) magnetic field and the magnetic fields inside each of the phases. To find the distribution of magnetic field between the two phases, one must solve in microscale the magnetic equations

$$\nabla \cdot \mathbf{B}^s = 0 \tag{79}$$

$$\nabla \times \mathbf{H}^s = 0 \tag{80}$$

for $\mathbf{x} \in \Omega^s$, and

$$\nabla \cdot \mathbf{B}^f = 0 \tag{81}$$

$$\nabla \times \mathbf{H}^f = 0 \tag{82}$$

for $\mathbf{x} \in \Omega^f$, with the interfacial boundary conditions

$$\mathbf{B}^s \cdot \mathbf{n}^s = -\mathbf{B}^f \cdot \mathbf{n}^f \tag{83}$$

$$\mathbf{H}_t^s = \mathbf{H}_t^f \tag{84}$$

for $\mathbf{x} \in S^i$, where the subscript t denotes the component tangential to the surface. The above equations are supplemented by a relation between \mathbf{H} and \mathbf{B} . Based on the standard assumption (Landau and Lifshitz, 1982), a linear relation is used

$$\mathbf{B}^s = \mu^s \mathbf{H}^s \tag{85}$$

$$\mathbf{B}^f = \mu^f \mathbf{H}^f \tag{86}$$

where μ^s and μ^f are, respectively, the magnetic permeability of the solid and fluid phase. The magnetic permeability is a material property that is generally dependent on the density for the fluid and the strain tensor for the solid. Solution of (79)–(86) subject to an external magnetic field, with a real porous medium geometry, is virtually impossible.

To overcome the difficulty, we shall take the traditional porous medium approach of using simple, conceptual models based on physical reasoning which can be validated by experimental observations. Consider a porous medium made of plane layers of two kinds of materials having isotropic magnetic permeabilities μ^s and μ^f . In the first model, we assume that the magnetic field is normal to the planes. It is clear from the above equations that the magnetic inductions are constant in all media :

$$\mathbf{B}^s = \mathbf{B}^f = \mathbf{B}_0 \tag{87}$$

which leads to a jump in \mathbf{H} ,

$$\mathbf{H}^s = \frac{\mathbf{B}_0}{\mu^s} \quad (88)$$

$$\mathbf{H}^f = \frac{\mathbf{B}_0}{\mu^f} \quad (89)$$

Define the volume averaged fields as

$$\tilde{\mathbf{H}} = (1 - \phi)\mathbf{H}^s + \phi\mathbf{H}^f \quad (90)$$

$$\tilde{\mathbf{B}} = (1 - \phi)\mathbf{B}^s + \phi\mathbf{B}^f \quad (91)$$

It is easily shown that an effective permeability

$$\mu^{\text{eff}} = \mu_v = \frac{\mu^s \mu^f}{(1 - \phi)\mu^f + \phi\mu^s} \quad (92)$$

exists such that

$$\tilde{\mathbf{B}} = \mu^{\text{eff}}\tilde{\mathbf{H}} \quad (93)$$

On the other hand, we assume in the second model that the magnetic field is parallel to the planes. Hence \mathbf{H} is continuous,

$$\mathbf{H}^s = \mathbf{H}^f = \mathbf{H}_0 \quad (94)$$

while \mathbf{B} has a jump

$$\mathbf{B}^s = \mu^s \mathbf{H}_0 \quad (95)$$

$$\mathbf{B}^f = \mu^f \mathbf{H}_0 \quad (96)$$

The effective permeability that satisfies (93) is

$$\mu^{\text{eff}} = \mu_h = (1 - \phi)\mu^s + \phi\mu^f \quad (97)$$

In fact we observe that (97) is an algebraic mean while (92) is a harmonic mean. It can be shown that $\mu_h \geq \mu_v$. Particularly, if we assume that $\mu^f \gg \mu^s$, (92) shows that $\mu^{\text{eff}} \approx \mu^s / (1 - \phi)$, while (97) gives $\mu^{\text{eff}} \approx \phi\mu^f$.

Now consider two types of composite material: a non-ferromagnetic solid containing isolated ferromagnetic inclusions, and a non-ferromagnetic porous matrix containing a penetrating and connected ferromagnetic fluid. The effective permeability model of (92) better describe the solid inclusion case while (97) is closer to the penetrating ferromagnetic field case. In fact, we can visualize these cases as a magnetic flux seeking paths of least magnetic resistance. For the solid inclusion case, the path is essentially blocked. It is obvious that the ferromagnetic fluid saturated porous media is a much more effective ferromagnetic material.

The above are conceptual models. A practical model needs to be built that is consistent with the conceptual models, yet with coefficients determined from laboratory test. This is introduced as follows. For an isotropic material we write

$$\mathbf{H}^s = \eta \mathbf{H}^f \quad (98)$$

where η is a partition coefficient dependent on the porous medium geometry. Following (90) we obtain

$$\bar{\mathbf{H}} = [\eta(1 - \phi) + \phi] \mathbf{H}^f \quad (99)$$

Consequently

$$\mathbf{H}^s = \beta_s \bar{\mathbf{H}} \quad (100)$$

$$\mathbf{H}^f = \beta_f \bar{\mathbf{H}} \quad (101)$$

where

$$\beta_s = \frac{\eta}{\eta(1 - \phi) + \phi} \quad (102)$$

$$\beta_f = \frac{1}{\eta(1 - \phi) + \phi} \quad (103)$$

are weighing factors. We note that if $\eta = 1$, then $\mathbf{H}^s = \mathbf{H}^f$ and we have model 2 (unconnected ferromagnetic material). If $\eta = \mu^f/\mu^s$, we have (88) and (89), which is model 1 (connected ferromagnetic material). In reality,

$$\frac{\mu^f}{\mu^s} \geq \eta \geq 1 \quad (104)$$

Given μ^s and μ^f , η is a function of porous media geometry and should be determined experimentally.

For determining η , we should rely on the measurement of \mathbf{B} , because it is the true mean field and \mathbf{H} is inferred from it. The effective value $\bar{\mathbf{B}}$ must be correctly defined such that it is consistent with its definition as the derivative of free energy, (44) and (45). We find

$$\bar{\mathbf{B}} = (1 - \phi)\beta_s \mathbf{B}^s + \phi\beta_f \mathbf{B}^f \quad (105)$$

This equation replaces (91). We observe that for the special case $\eta = 1$, (91) is satisfied, and so does for the case $\eta = \mu^f/\mu^s$. If we know the magnetic induction of each phase, and can measure the effective value, η can be determined from (105).

We can now rewrite terms involving magnetic field of individual phases, as in (68), (69), (75) and (78), in terms of the average magnetic field:

$$B_i^s H_j^s = \beta_s B_i^s \bar{H}_j \quad (106)$$

$$B_i^f H_j^f = \beta_f B_i^f \bar{H}_j \quad (107)$$

and

$$p_H^s = \beta_s \left(\int_0^R \mathbf{M}^s \cdot d\mathbf{H} + \int_0^R \frac{\partial \mathbf{M}^s}{\partial e^s} \cdot d\mathbf{H} \right) + \frac{1}{8\pi} \beta_s^2 \mathbf{H} \cdot \mathbf{H} \quad (108)$$

$$p_H^f = \beta_f \int_0^R \frac{\partial(\mathbf{M}^f/\rho_f)}{\partial(1/\rho_f)} \cdot d\mathbf{H} + \frac{1}{8\pi} \beta_f^2 \mathbf{H} \cdot \mathbf{H} \quad (109)$$

9. MAGNETIC SUSCEPTIBILITY

The magnetization is related to the magnetic field as

$$\mathbf{M} = \chi \mathbf{H} \quad (110)$$

where χ is known as the magnetic susceptibility. In a stationary case the magnetic susceptibility of a ferromagnetic liquid is proportional to the number concentration of magnetic particles, n , and their individual magnetic moment, m_H :

$$\chi = nm_H R(\mathbf{H}, T) \quad (111)$$

where the coefficient R depends on the magnetic field and the temperature, because the temperature defines the degree of disordering of magnetic moments of the individual particles. However, there are a number of mechanisms that can modify the stationary equation. First, there are two kinds of relaxation processes: a fast process associated with the orientation of particles along the direction of a local magnetic field, and a slow process associated with the equalization of concentration of particles which is initially inhomogeneous in space. The characteristic time of the first relaxation processes is extremely small (Takeomi and Tickatsumi, 1988), about 10^{-9} s, and can be ignored when our interest lies in mechanical processes. This is however not so for the slow process which smooths out concentration inhomogeneities. In the general case, one has to add an equation of evolution of local concentration of the magnetic particles. It is a diffusion equation corrected for the interaction of particle with a magnetic field. In most cases of interest, the mechanical processes are relatively fast so this diffusion process does not have time to come into play. This leaves us only the mechanical process itself to consider. For this process we may assume that the particle concentration is proportional to the density of liquid. As the density increases due to mechanical deformation, so does the particle concentration. We hence write

$$\chi = \frac{\rho_f}{\rho_f^0} n_0 m_H R(\mathbf{H}, T) \quad (112)$$

where ρ_f^0 is the initial fluid density, and n_0 is the initial particle number concentration. In this case, the integral in (109) vanishes, and it simplifies to

$$p_H^f = \frac{1}{8\pi} \beta_f^2 \mathbf{H} \cdot \mathbf{H} \quad (113)$$

10. MECHANICAL EQUILIBRIUM

Using the definitions (72) and (76), we can sort (68) and (69) into the following:

$$[(1 - \phi A_s) p_m^s + \phi A_s p_m^f] \delta_{ij} - C_s \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} = -\sigma_{ij}^s + \frac{1}{4\pi} B_i^s H_j^s - [(1 - \phi A_s) p_H^s + \phi A_s p_H^f] \delta_{ij} \quad (114)$$

$$\begin{aligned}
 & [(1 + A_f - \phi A_f)p_m^f - (1 - \phi)A_f p_m^s] \delta_{ij} - \frac{1 - \phi}{\phi} C_f \frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} \\
 & = -\sigma_{ij}^f + \frac{1}{4\pi} B_i^f H_j^f - [(1 + A_f - \phi A_f)p_m^f - (1 - \phi)A_f p_m^s] \delta_{ij} \quad (115)
 \end{aligned}$$

We observe that the left sides of (114) and (115) are of mechanical origin as they contain mechanical arguments $e_f, e^s, \langle e_{ij}^s \rangle$, mechanical equations of state $p_m^s(e^s, \langle e_{ij}^s \rangle)$ and $p_m^f(\rho_f)$, and material coefficients A_s and A_f . We note that $A_s(\rho_f, e^s, \langle e_{ij}^s \rangle, \mathbf{H}^s(\mathbf{H}), \mathbf{H}^f(\mathbf{H}))$ and $A_f(\rho_f, e^s, \langle e_{ij}^s \rangle, \mathbf{H}^s(\mathbf{H}), \mathbf{H}^f(\mathbf{H}))$ are functions of magnetic field. However, in the absence of such a field, these coefficients approach their mechanical limits. In this form, (114) and (115) help to distinguish the impact of magnetic forces (presented in the right sides) and the impact of magnetic field on material parameters. Equations (114) and (115) together with (50) and (51), which relate internal strains to external (apparent) strains, are the constitutive equations that associate stresses with strains.

It is necessary to add to this system the proper equations of magnetic field :

$$\nabla \cdot \mathbf{B} = 0 \quad (116)$$

$$\nabla \times \mathbf{H} = 0 \quad (117)$$

with proper initial and boundary conditions.

Next, consider force equilibrium. The resultant forces per unit volume of solid and fluid phases due to surface stresses, Q_s and Q_f , can be found by integrating the stresses over the solid and fluid surfaces, S^s and S^f , respectively

$$Q_s = \frac{1}{\Omega^s} \oint_{S^s} \sigma_{ij}^s n_j dS = \frac{1}{\Omega^s} \int_{\Omega^s} \sigma_{ij,j}^s d\Omega \quad (118)$$

$$Q_f = \frac{1}{\Omega^f} \oint_{S^f} \sigma_{ij}^f n_j dS = \frac{1}{\Omega^f} \int_{\Omega^f} \sigma_{ij,j}^f d\Omega \quad (119)$$

where we have applied the divergence theorem that converts surface integrals to volume ones. The total resultant force per unit REV is

$$Q = (1 - \phi)Q_s + \phi Q_f \quad (120)$$

These average forces, adjusted from phase volume to total volume, must be equated to the total volume averaged body force, inertia force, and interaction force between phases, leading to the differential form of equilibrium equations :

$$(1 - \phi)\dot{\sigma}_{ij,j}^s = (1 - \phi)\rho_s \ddot{u}_i^s - (1 - \phi)\rho_s g_i - I_i \quad (121)$$

$$\phi \dot{\sigma}_{ij,j}^f = \phi \rho_f \ddot{u}_i^f - \phi \rho_f g_i + I_i \quad (122)$$

where the overdots are used to denote differentiation with respect to time, g_i is the gravitational vector component, and I_i is the interaction force per unit volume. It is of interest to remark that the derivation leading to (121) and (122) is valid for both homogeneous and inhomogeneous (ϕ being a function of space) materials. This is contrary to an earlier presentation (Nikolaevskiy, 1984) which indicated the existence of an interaction force resulting from the variation of porosity.

Equations (121) and (122) can be summed to give the equilibrium equation of combined phases

$$\sigma_{ij,j} = \rho \ddot{u}_i^s + \phi \rho_f \ddot{w}_i - \rho g_i \quad (123)$$

where $\sigma_{ij} = (1 - \phi)\sigma_{ij}^s + \phi\sigma_{ij}^f$ is the total stress tensor and $\rho = (1 - \phi)\rho_s + \phi\rho_f$ is the bulk density.

The interaction force is created by the relative motion between the solid and fluid phase and consists of two parts

$$I_i = \rho_a \ddot{w}_i + b \dot{w}_i \quad (124)$$

where ρ_a is the added mass density (Biot, 1956a) attributed to the momentum transfer between solid and fluid, b is a resistivity coefficient, and

$$w_i = u_i^f - u_i^s \quad (125)$$

is the relative fluid-solid displacement. In fact, under general condition the interaction force should be written as $I = Y\{\dot{w}_i\}$ where Y is a differential operator of time such that it incorporates the fluid inertia, viscosity, apparent (frequency-dependent) viscosity, and other fluid rheological effects (Biot, 1962).

11. LINEARIZED MECHANICAL MODEL

For a better understanding of the structure of the present theory, let us consider its linearized variants. We can consider two kinds of linearization. The first kind is a partial linearization which considers the medium to be linear from the mechanical point-of-view. The problem remains nonlinear due to the magnetic field. The second kind of linearization is a complete linearization in which the magnetic parts of the equations are also linearized. This requires that the changes of magnetic field due to medium deformation or other causes be small. We shall consider the mechanical linearization first.

Under linear assumption, we can write the following constitutive relations :

$$p_m^s = -K_s e^s \quad (126)$$

$$p_m^f = -K_f e^f \quad (127)$$

where K_s and K_f are, respectively, the bulk modulus of solid and fluid phase. In the above we have denoted $\delta\rho_f/\rho_f = -\delta e^f$, and then dropped the variation notation δ in δe^s and δe^f for consistency with common notation. From the free energy term we obtain

$$\frac{\partial F^s}{\partial \langle e_{ij}^s \rangle} = 2G_s \langle e_{ij}^s \rangle = 2G_s e_{ij}^s - \frac{2}{3} G_s \delta_{ij} e^s \quad (128)$$

where G_s is the shear modulus of the solid phase, and we have used the definition in (43). Substituting (126) to (128) into (114) and (115), and dropping for the moment the magnetic terms, we obtain

$$\sigma_{ij}^s = \left[(1 - \phi_0 A_s) K_s - \frac{2}{3} G_s C_s \right] \delta_{ij} e^s + \phi_0 A_s K_f \delta_{ij} e^f + 2G_s C_s e_{ij}^s \quad (129)$$

$$\sigma_{ij}^f = (1 + A_f - \phi_0 A_f) K_f \delta_{ij} e^f - (1 - \phi_0) A_f K_s \delta_{ij} e^s \quad (130)$$

where the fluid is assumed to be Newtonian, hence we have set $C_f = 0$, and the porosity ϕ is replaced by its initial value ϕ_0 .

To model the bulk continuum behavior, the internal strains in (129) and (130) must be converted to the external ones, which are the apparent strains of the REV. Utilizing (50), (51) and (55), the above equations become

$$\sigma_{ij}^s = (a^{ss}\varepsilon^s + a^{sf}\varepsilon^f)\delta_{ij} + a^{sd}\varepsilon_{ij}^s \tag{131}$$

$$\sigma_{ij}^f = (a^{fs}\varepsilon^s + a^{ff}\varepsilon^f)\delta_{ij} \tag{132}$$

where the coefficients are

$$a^{ss} = (1 - \phi_0 A_s)^2 K_s + \phi_0 (1 - \phi_0) A_s^2 K_f - \frac{2}{3} C_s (1 - \phi_0 A_s) G_s \tag{133}$$

$$a^{sf} = -\phi_0 (1 - \phi_0 A_s) A_f K_s + \phi_0 (1 + A_f - \phi_0 A_f) A_s K_f \tag{134}$$

$$+ \frac{2}{3} \phi_0 C_s A_f G_s \tag{135}$$

$$a^{fs} = -(1 - \phi_0)(1 - \phi_0 A_s) A_f K_s + (1 - \phi_0)(1 + A_f - \phi_0 A_f) A_s K_f \tag{136}$$

$$a^{ff} = \phi_0 (1 - \phi_0) A_f^2 K_s + (1 + A_f - \phi_0 A_f)^2 K_f \tag{137}$$

$$a^{sd} = 2C_s^2 G_s \tag{138}$$

Equation (132) can be contracted to obtain a (negative) fluid pressure $\sigma^f = \frac{1}{3} \sigma_{ii}^f$, hence

$$\sigma^f = a^{fs}\varepsilon^s + a^{ff}\varepsilon^f \tag{139}$$

We note that (131) and (139) are now at the same level as the linear poroelasticity theory of Biot (1941).

12. LINEARIZED MAGNETIC MODEL

Next, consider the case of complete linearization. In the presence of a magnetic field, we need to take into account the existence of an initial stress and a stationary magnetic field. The stress and magnetic quantities sought for are perturbations from their initial states. Assume that the initial magnetic field is $\tilde{\mathbf{H}}^0$, and the perturbation $\tilde{\mathbf{H}}$ from it is a small quantity. Following (113), we note that the magnetic pressure can be expressed in the linearized form :

$$p_H^f = \frac{1}{8\pi} \beta_f^2 \tilde{H}_i^0 \tilde{H}_i^0 + \frac{1}{4\pi} \beta_f^2 \tilde{H}_i^0 \tilde{H}_i \tag{140}$$

Also, we can write in the first-order :

$$\begin{aligned} B_i^f H_j^f &= B_i^{f0} H_j^{f0} + \tilde{B}_i^f H_j^{f0} + B_i^{f0} \tilde{H}_j^f \\ &= B_i^{f0} H_j^{f0} + \beta_f^2 \tilde{H}_j^0 \tilde{H}_i + \beta_f^2 \tilde{H}_i^0 \tilde{H}_j + 4\pi\beta_f M_i^{f0} \tilde{H}_j + 4\pi\beta_f \tilde{H}_j^0 \tilde{M}_i^f \end{aligned} \tag{141}$$

in which we have utilized (74) and (101) and used the notation $(\)^0$ to denote initial value, and (\sim) for perturbed quantities. In accordance with (110) and (112), we find

$$M_i^{f0} = \beta_f n_0 m_H R^{f0} \tilde{H}_i^0 \tag{142}$$

where $R^{f0} = R^f(\mathbf{H}^{f0}, T)$. From the perturbation of (110) and (112) we obtain

$$\begin{aligned}
\tilde{M}_i^f &= \beta_f n_0 m_H (R^0 \tilde{H}_i + \beta_f \partial R_f^0 \tilde{H}_i^0 \tilde{H}_j - R^0 \tilde{H}_i^0 e^f) \\
&= \beta_f n_0 m_H [R^0 \tilde{H}_i + \beta_f \partial R_f^0 \tilde{H}_i^0 \tilde{H}_j \\
&\quad - (1 - \phi_0) R^0 \tilde{H}_i^0 A_s \varepsilon^s - R^0 \tilde{H}_i^0 (1 + A_f - \phi_0 A_f) \varepsilon^f]
\end{aligned} \tag{143}$$

in which

$$\partial R_f^0 = \left. \frac{\partial R^f(\mathbf{H}^f, T)}{\partial H_j^f} \right|_{H_j^f = \beta_f \tilde{H}_j^0} \tag{144}$$

After substitution of (140) to (143) into (114) and (115), and dropping the initial states, we can present these equations in the form:

$$\sigma_{ij}^s = (\alpha^{ss} \varepsilon^s + \alpha^{sf} \varepsilon^f - b_k^s \tilde{H}_k) \delta_{ij} + \alpha^{sd} \varepsilon_{ij}^s \tag{145}$$

$$\sigma_{ij}^f = (\alpha^{fs} \varepsilon^s + \alpha^{ff} \varepsilon^f - b_k^f \tilde{H}_k) \delta_{ij} - b_{ij}^{fs} \varepsilon^s - b_{ij}^{ff} \varepsilon^f + b_j^{fh} \tilde{H}_i + b_i^{fh} \tilde{H}_j + b_{ijk} \tilde{H}_k \tag{146}$$

where

$$b_k^s = \frac{1}{4\pi} \phi_0 \beta_f^2 A_s \tilde{H}_k^0 \tag{147}$$

$$b_k^f = \frac{1}{4\pi} \beta_f^2 (1 + A_f - \phi_0 A_f) \tilde{H}_k^0 \tag{148}$$

$$b_{ij}^{fs} = (1 - \phi_0) \beta_f^2 n_0 m_H R^0 A_s \tilde{H}_i^0 \tilde{H}_j^0 \tag{149}$$

$$b_{ij}^{ff} = \beta_f^2 n_0 m_H R^0 (1 + A_f - \phi_0 A_f) \tilde{H}_i^0 \tilde{H}_j^0 \tag{150}$$

$$b_i^{fh} = \frac{1}{4\pi} \beta_f^2 (1 + 4\pi n_0 m_H R^0) \tilde{H}_i^0 \tag{151}$$

$$b_{ijk} = \beta_f^3 n_0 m_H \partial R_k^0 \tilde{H}_i^0 \tilde{H}_j^0 \tag{152}$$

It should be noted that in the above we assumed the solid phase to be magnetically inert such that the solid part of magnetic pressure and stress does not exist. We observe in (145) and (146) that even if the structural coefficients A_s , A_f , etc. are assumed to be independent of the magnetic field, still the effective modulus of medium manifests such dependence, because of the additional ‘‘magnetic solidity’’. We also note that despite the fluid is considered Newtonian, shear stress exists in the fluid phase due to the fact that the magnetic field is a vector field.

We assume that at the initial state the medium is in mechanical equilibrium such that we only need to consider the perturbed terms as shown above. These constitutive equations (145) and (146) can be substituted into the equilibrium equations (121) and (122) to eliminate the stress expressions. To form a complete solution system, the equations of the magnetic field must be introduced and linearized. Based on (74), (116), (117) and (143), we find

$$\nabla \cdot (c_i^h \tilde{\mathbf{H}}^0 \tilde{H}_i + c \tilde{\mathbf{H}} - c^s \tilde{\mathbf{H}}^0 \varepsilon^s - c^f \tilde{\mathbf{H}}^0 \varepsilon^f) = 0 \tag{153}$$

$$\nabla \times \tilde{\mathbf{H}} = 0 \tag{154}$$

in which

$$c^s = 4\pi(1 - \phi_0) \beta_f n_0 m_H R^0 A_s \tag{155}$$

$$c^f = 4\pi\beta_f n_0 m_H R^0 (1 + A_f - \phi_0 A_f) \quad (156)$$

$$c_i^h = 4\pi\beta_f^2 n_0 m_H \partial R_i^{f_0} \quad (157)$$

$$c = \beta_f (1 + 4\pi n_0 m_H R^0) \quad (158)$$

We observe from (145), (146) and (153) that these equations are coupled. The system (121), (122), (145), (146), (153) and (154) requires associated boundary conditions in mechanical variables as well as for a magnetic field. It is then suitable for a mathematical solution.

13. CONCLUSION

We have constructed, following thermodynamic considerations of virtual displacement and free energy, the continuum mechanics model of a porous medium saturated with a ferromagnetic fluid. Although the mechanics of poroelastic medium has been traditionally constructed through phenomenological approach, such as that of Biot (1941, 1962) and others (Detournay and Cheng, 1993; Nikolaevskiy, 1996), the current thermodynamic construction is essential for the following reasons:

- (1) its ability to model irreversible deformation due to permanent pore structure changes;
- (2) its ability to model incrementally nonlinear behavior of porous medium; and
- (3) the use of internal and external strain variables and their role in external work and internal energy allows us to correctly model the coupling of the mechanical and the magnetic field.

To our knowledge, no such model, linear or nonlinear, has previously been constructed. It becomes useful to provide a linearized, more tractable version of the general theory. Such a model is presented. It is demonstrated that in the absence of a magnetic field, the linearized model is consistent with the Biot theory.

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